CSCE 222
Discrete Structures for Computing

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Relations

Chapter 9
Relations

Given two sets $A$ and $B$:
A relation $R$ from $A$ to $B$ is a subset of $A \times B$.

\[ R \subseteq A \times B \]

Given an ordered pair $(x,y)$ in $A \times B$:
x is related to $y$ by $R$ iff $(x,y)$ is in $R$

$A$ is the domain of $R$ and $B$ is the co-domain of $R$

Some Relations on Integers

Here are some relations on $Z$:

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b\ \text{or} \ a = -b\}$

$R_4 = \{(a,b) \mid a = b\}$

$R_5 = \{(a,b) \mid a = b + 1\}$

$R_6 = \{(a,b) \mid a + b \leq 3\}$
Number of relations

How many relations are there on a set \( A \) with \( n \) elements? Each relation is a subset of \( A \times A \).

How big is \( A \times A \)? \(|A \times A| = n^2\)

How many subsets are there of a set of size \( n^2 \)?
The final answer is \( 2^{n^2} \).

Properties of Relations

These are useful properties that some (but not all) relations have:

- Reflexive
- Symmetric
- Antisymmetric
- Transitive

Let’s see what they are...
Reflexive Relation

A relation \( R \) on set \( A \) is reflexive if \((a, a) \in R\) for all \( a \in A\)

Activity

Which of the following relations are reflexive? Why?

\[ R_1 = \{(a,b) | a \leq b\} \]
\[ R_2 = \{(a,b) | a > b\} \]
\[ R_3 = \{(a,b) | a = b \text{ or } a = -b\} \]
\[ R_4 = \{(a,b) | a = b\} \]
\[ R_5 = \{(a,b) | a = b + 1\} \]
\[ R_6 = \{(a,b) | a + b \leq 3\} \]
Symmetric Relation

A relation $R$ on set $A$ is symmetric if $(a,b) \in R$ implies $(b,a) \in R$

Activity

Which of the following relations are symmetric? Why?

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a,b) \mid a = b\}$

$R_5 = \{(a,b) \mid a = b + 1\}$

$R_6 = \{(a,b) \mid a + b \leq 3\}$
Antisymmetric Relation

A relation R on set A is antisymmetric if \( (a,b) \in R \) and \( (b,a) \in R \) implies \( a = b \).

Activity

Which of the following relations are antisymmetric? Why?

- \( R_1 = \{ (a,b) \mid a \leq b \} \)
- \( R_2 = \{ (a,b) \mid a > b \} \)
- \( R_3 = \{ (a,b) \mid a = b \text{ or } a = -b \} \)
- \( R_4 = \{ (a,b) \mid a = b \} \)
- \( R_5 = \{ (a,b) \mid a = b + 1 \} \)
- \( R_6 = \{ (a,b) \mid a + b \leq 3 \} \)
Transitive Relations

A relation $R$ on set $A$ is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

Composing Relations

Another way to combine relations is analogous to function composition.

Suppose $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$.

The composite of $R$ and $S$ is the relation from $A$ to $C$ consisting of ordered pairs $(a, c)$ such that there exists $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$.

Notation: $S \circ R$
Composing Relations Example #1

A = {1,2,3}
B = {1,2,3,4}
C = {0,1,2}

Suppose:
R = {(1,1), (1,4), (2,3), (3,1), (3,4)}
S = {(1,0), (2,0), (3,1), (3,2), (4,1)}
Then S ◦ R = {(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}.

Composing Relations Example #2

Suppose R is the relation on the set of people such that (a, b) ∈ R iff a is a parent of b.
Then R ∘ R consists of pairs (a, c) such that there is a person b where a is a parent of b, and b is a parent of c.
i.e., R ∘ R is the “grandparent” relation.
Equivalence Relations

A relation $R$ on a set $A$ is an equivalence relation if it is:

1. reflexive
2. symmetric
3. transitive

Notation: if $(a, b) \in R$, then we write $a \sim b$.

Equivalence Relations examples

- All pairs $(a, b)$ of integers where $a = b$ or $a = -b$
- All pairs $(a, b)$ of real numbers where $a - b$ is an integer
- All pairs $(a, b)$ of integers that have the same remainder when divided by a fixed integer $m$
- All pairs of strings over some alphabet with the same length
Directed Graphs

A directed graph is a set $V$ of vertices and a set $E$ of edges (a subset of $V \times V$)

An edge of the form $(a, a)$ is called a loop.

Directed Graph example

Directed graph with vertices $a$, $b$, $c$, and $d$, and edges $(a, b)$, $(a, d)$, $(b, b)$, $(b, d)$, $(c, a)$, $(c, b)$, and $(d, b)$:
Activity

What are the ordered pairs in the relation represented by this directed graph?

Directed Graph and Relation Properties

When $A = B$, the directed graph representation gives a visual way to check for certain properties of the relation:

**Reflexive:** every vertex has a self-loop

**Symmetric:** if $(x, y)$ is an edge, then so is $(y, x)$

**Antisymmetric:** if $(x, y)$ is an edge, then $(y, x)$ is not an edge

**Transitive:** if $(x, y)$ and $(y, z)$ are edges, then $(x, z)$ is an edge (“complete the triangle”)

Activity

The directed graph for relation $R$ on set $A$:
$A = \{2, 3, 4, 6, 7, 9\}$
For all $x, y \in A$, $x \, R \, y \iff 3 \mid (x - y)$ is:

Reflexivity

Is the relation $R$ is reflexive?
Is the relation $R$ symmetric?

Is the relation $R$ transitive?
End