Relations

Given two sets A and B:
A relation $R$ from $A$ to $B$ is a subset of $A \times B$.

$R \subseteq A \times B$

Given an ordered pair $(x,y)$ in $A \times B$:
x is related to y by $R$ iff $(x,y)$ is in $R$

A is the domain of $R$ and B is the co-domain of $R$

Some Relations on Integers

Here are some relations on $\mathbb{Z}$:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$
Number of relations

How many relations are there on a set $A$ with $n$ elements?
(Each relation is a subset of $A \times A$)

1. How big is $A \times A$?
   
   $|A \times A| = n^2$

2. How many subsets are there of a set of size $n^2$?
   
   $2^{n^2}$ subsets

Properties of Relations

Useful properties that some (but not all) relations have:

- Reflexive
- Symmetric
- Antisymmetric
- Transitive

Let’s see what they are...
Reflexive Relation

• Relation $R$ on set $A$ is reflexive if: $\forall a \in A$, $(a, a) \in R$

• $\forall a \in A$, $a R a$

Activity

Which of the following relations are reflexive? Why?

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b$ or $a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$
Activity

Which of the following relations are reflexive? Why?

$R_1 = \{(a, b) \mid a \leq b\}$ is reflexive

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ is reflexive

$R_4 = \{(a, b) \mid a = b\}$ is reflexive

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$

Symmetric Relation

• Relation $R$ on set $A$ is symmetric iff $\forall a, b \in A,$
  if $(a, b) \in R$ then $(b, a) \in R$

• $\forall a, b \in A,$ if $a R b$ then $b R a$
Activity

Which of the following relations are symmetric? Why?

\[ R_1 = \{(a, b) \mid a \leq b\} \]
\[ R_2 = \{(a, b) \mid a > b\} \]
\[ R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \]
\[ R_4 = \{(a, b) \mid a = b\} \]
\[ R_5 = \{(a, b) \mid a = b + 1\} \]
\[ R_6 = \{(a, b) \mid a + b \leq 3\} \]

is symmetric

is symmetric

is symmetric
Antisymmetric Relation

Relation \( R \) on set \( A \) is antisymmetric if \((a, b) \in R\) and \((b, a) \in R\) implies \( a = b \).

Notes:
- The terms symmetric and antisymmetric are not opposites.
- A relation can have both properties or lack both properties.

Activity

Which of the following relations are antisymmetric? Why?

\( R_1 = \{(a, b) \mid a \leq b\} \)

\( R_2 = \{(a, b) \mid a > b\} \)

\( R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \)

\( R_4 = \{(a, b) \mid a = b\} \)

\( R_5 = \{(a, b) \mid a = b + 1\} \)

\( R_6 = \{(a, b) \mid a + b \leq 3\} \)
Activity

Which of the following relations are antisymmetric? Why?

\[
R_1 = \{(a, b) \mid a \leq b\} \text{ is antisymmetric}
\]

\[
R_2 = \{(a, b) \mid a > b\} \text{ is antisymmetric}
\]

\[
R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}
\]

\[
R_4 = \{(a, b) \mid a = b\} \text{ is antisymmetric}
\]

\[
R_5 = \{(a, b) \mid a = b + 1\} \text{ is antisymmetric}
\]

\[
R_6 = \{(a, b) \mid a + b \leq 3\}
\]

Transitive Relations

• Relation \( R \) on set \( A \) is transitive iff \( \forall a, b, c \in A, \) if \( (a, b) \in R \) and \( (b, c) \in R \) then \( (a, c) \in R \)

• \( \forall a, b, c \in A, \) if \( a \mathrel{R} b \) and \( b \mathrel{R} c \), then \( a \mathrel{R} c \)
Which of the following relations are transitive? Why?

\[ R_1 = \{(a, b) \mid a \leq b\} \]

\[ R_2 = \{(a, b) \mid a > b\} \]

\[ R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \]

\[ R_4 = \{(a, b) \mid a = b\} \]

\[ R_5 = \{(a, b) \mid a = b + 1\} \]

\[ R_6 = \{(a, b) \mid a + b \leq 3\} \]
Composing Relations

Another way to combine relations is analogous to function composition.

Suppose $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$.

The composite of $R$ and $S$ is the relation from $A$ to $C$ consisting of ordered pairs $(a, c)$ such that $\exists b \in B$ with $(a, b) \in R$ and $(b, c) \in S$.

Notation: $S \circ R$

Composing Relations example 1

$A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4\}$
$C = \{0, 1, 2\}$

Suppose:
$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$
$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$

Then $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$. 
Composing Relations example 2

Suppose $R$ is the relation on the set of people such that:

$$(a, b) \in R \text{ iff } a \text{ is the parent of } b.$$

Then $R \circ R$ consists of pairs $(a, c)$ such that there is a person $b$ where:

- $a$ is the parent of $b$
- $b$ is the parent of $c$

i.e., $R \circ R$ is the grandparent relation.

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Equivalence Relations

A relation $R$ on a set $A$ is an equivalence relation if it is:

1. reflexive
2. symmetric
3. transitive

Notation: if $(a, b) \in R$, then we write $a \sim b$. 
Equivalence Relations examples

• All pairs \((a, b)\) of integers where \(a = b\) or \(a = -b\)

• All pairs \((a, b)\) of real numbers where \(a - b\) is an integer

• All pairs \((a, b)\) of integers that have the same remainder when divided by a fixed integer \(m\)

• All pairs of strings over some alphabet with the same length

Directed Graphs

A directed graph is a set \(V\) of vertices and a set \(E\) of edges \((a\) subset of \(V \times V)\)

An edge of the form \((a, a)\) is called a loop.
Directed Graph example

Vertices: \{a, b, c, d\}
Edges: \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}

Example

What are the ordered pairs in the relation represented by this directed graph?
Directed Graph and Relation Properties

When \( A = B \), the directed graph gives a visual way to check for certain properties of the relation:

- **Reflexive**: every vertex has a self-loop
- **Symmetric**: if \((x, y)\) is an edge, then so is \((y, x)\)
- **Antisymmetric**: if \((x, y)\) is an edge, then \((y, x)\) is not an edge
- **Transitive**: if \((x, y)\) and \((y, z)\) are edges, then \((x, z)\) is an edge (complete the triangle)

**Example**

The directed graph for relation \( R \) on set \( A \):

\( A = \{2, 3, 4, 6, 7, 9\} \)

For all \( x, y \in A \), \( x R y \iff 3 \mid (x - y) \) is:
Reflexivity

Is the relation R is reflexive?

Symmetry

Is the relation R symmetric?
Transitivity

Is the relation R transitive?

End