Relations

Given two sets A and B:

A relation R from A to B is a subset of A×B.

\[ R \subseteq A \times B \]

Given an ordered pair (x,y) in A×B:

x is related to y by R iff (x,y) is in R

A is the domain of R and B is the co-domain of R

Some Relations on Integers

Here are some relations on \( \mathbb{Z} \):

\[ R_1 = \{(a, b) \mid a \leq b\} \]

\[ R_2 = \{(a, b) \mid a > b\} \]

\[ R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \]

\[ R_4 = \{(a, b) \mid a = b\} \]

\[ R_5 = \{(a, b) \mid a = b + 1\} \]

\[ R_6 = \{(a, b) \mid a + b \leq 3\} \]
Number of relations

How many relations are there on a set $A$ with $n$ elements?
(Each relation is a subset of $A \times A$)

1. How big is $A \times A$?
$|A \times A| = n^2$

2. How many subsets are there of a set of size $n^2$?
$2^{n^2}$ subsets

Properties of Relations

Useful properties that some (but not all) relations have:

- Reflexive
- Symmetric
- Antisymmetric
- Transitive

Let’s see what they are ...
Reflexive Relation

• Relation $R$ on set $A$ is reflexive if: $\forall a \in A$, $(a, a) \in R$

• $\forall a \in A$, $a R a$

Activity 7

Which of the following relations are **reflexive**? Why?

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$
Activity 7 solution

Which of the following relations are reflexive? Why?

$R_1 = \{(a, b) \mid a \leq b\}$ is reflexive

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ is reflexive

$R_4 = \{(a, b) \mid a = b\}$ is reflexive

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$

Symmetric Relation

- Relation $R$ on set $A$ is symmetric iff $\forall a, b \in A,$
  
  if $(a, b) \in R$ then $(b, a) \in R$

- $\forall a, b \in A,$ if $a R b$ then $b R a$
Activity 7

Which of the following relations are symmetric? Why?

\[ R_1 = \{(a, b) \mid a \leq b\} \]
\[ R_2 = \{(a, b) \mid a > b\} \]
\[ R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \]
\[ R_4 = \{(a, b) \mid a = b\} \]
\[ R_5 = \{(a, b) \mid a = b + 1\} \]
\[ R_6 = \{(a, b) \mid a + b \leq 3\} \]

Activity 7 solution

Which of the following relations are symmetric? Why?

\[ R_1 = \{(a, b) \mid a \leq b\} \]
\[ R_2 = \{(a, b) \mid a > b\} \]
\[ R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \text{ is symmetric} \]
\[ R_4 = \{(a, b) \mid a = b\} \text{ is symmetric} \]
\[ R_5 = \{(a, b) \mid a = b + 1\} \]
\[ R_6 = \{(a, b) \mid a + b \leq 3\} \text{ is symmetric} \]
Antisymmetric Relation

• Relation $R$ on set $A$ is antisymmetric if $\forall a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.
• $\forall a, b \in A$, if $a R b$ and $b R a$ then $a = b$.

Notes:
• The terms symmetric and antisymmetric are not opposites.
• A relation can have both properties or lack both properties.

Activity 7

Which of the following relations are antisymmetric? Why?

$R_1 = \{(a, b) \mid a \leq b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \leq 3\}$
Activity 7 solution

Which of the following relations are antisymmetric? Why?

- $R_1 = \{(a, b) \mid a \leq b\}$ is antisymmetric
- $R_2 = \{(a, b) \mid a > b\}$ is antisymmetric
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a, b) \mid a = b\}$ is antisymmetric
- $R_5 = \{(a, b) \mid a = b + 1\}$ is antisymmetric
- $R_6 = \{(a, b) \mid a + b \leq 3\}$

Transitive Relations

- Relation $R$ on set $A$ is transitive iff $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

- $\forall a, b, c \in A$, if $a \, R \, b$ and $b \, R \, c$, then $a \, R \, c$
Activity 7

Which of the following relations are transitive? Why?

\( R_1 = \{(a, b) \mid a \leq b\} \)

\( R_2 = \{(a, b) \mid a > b\} \)

\( R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \)

\( R_4 = \{(a, b) \mid a = b\} \)

\( R_5 = \{(a, b) \mid a = b + 1\} \)

\( R_6 = \{(a, b) \mid a + b \leq 3\} \)

Activity 7 solution

Which of the following relations are transitive? Why?

\( R_1 = \{(a, b) \mid a \leq b\} \text{ is transitive} \)

\( R_2 = \{(a, b) \mid a > b\} \text{ is transitive} \)

\( R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \text{ is transitive} \)

\( R_4 = \{(a, b) \mid a = b\} \text{ is transitive} \)

\( R_5 = \{(a, b) \mid a = b + 1\} \)

\( R_6 = \{(a, b) \mid a + b \leq 3\} \)
Another way to combine relations is analogous to function composition.

Suppose $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$. The composite of $R$ and $S$ is the relation from $A$ to $C$ consisting of ordered pairs $(a, c)$ such that $\exists b \in B$ with $(a, b) \in R$ and $(b, c) \in S$.

Notation: $S \circ R$

Composing Relations example 1

$A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4\}$
$C = \{0, 1, 2\}$

Suppose:
$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$
$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$

Then $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$. 
Suppose $R$ is the relation on the set of people such that:

$$(a, b) \in R \iff a \text{ is the parent of } b.$$ 

Then $R \circ R$ consists of pairs $(a, c)$ such that there is a person $b$ where:

- $a$ is the parent of $b$
- $b$ is the parent of $c$

i.e., $R \circ R$ is the \textit{grandparent} relation.

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A relation $R$ on a set $A$ is an \textit{equivalence} relation if it is:

1. reflexive
2. symmetric
3. transitive

Notation: if $(a, b) \in R$, then we write $a \sim b$. 
Equivalence Relations examples

- All pairs \((a, b)\) of integers where \(a = b\) or \(a = -b\)
- All pairs \((a, b)\) of real numbers where \(a - b\) is an integer
- All pairs \((a, b)\) of integers that have the same remainder when divided by a fixed integer \(m\)
- All pairs of strings over some alphabet with the same length

Directed Graphs

A directed graph is a set \(V\) of vertices and a set \(E\) of edges (a subset of \(V \times V\))

An edge of the form \((a, a)\) is called a loop.
Directed Graph example

Vertices: \{a, b, c, d\}
Edges: \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}

Example

What are the ordered pairs in the relation represented by this directed graph?
Directed Graph and Relation Properties

When \( A = B \), the directed graph gives a visual way to check for certain properties of the relation:

**Reflexive:** every vertex has a self-loop

**Symmetric:** if \((x, y)\) is an edge, then so is \((y, x)\)

**Antisymmetric:** if \((x, y)\) is an edge, then \((y, x)\) is not an edge

**Transitive:** if \((x, y)\) and \((y, z)\) are edges, then \((x, z)\) is an edge (complete the triangle)

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**Example**

The directed graph for relation \( R \) on set \( A \):

\[ A = \{2, 3, 4, 6, 7, 9\} \]

For all \( x, y \in A \), \( x R y \iff 3 \mid (x - y) \) is:
Is the relation $R$ is reflexive?

Is the relation $R$ symmetric?
Transitivity

Is the relation $R$ transitive?