CSCE 222
Discrete Structures for Computing

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Solving Recurrences

Chapter 8
Recurrence

A recurrence relation is a rule for determining elements of a sequence using preceding elements of the sequence.

Applications:
• Solve counting problems in general
• Model the running time of certain algorithms

It is useful to know how to solve recurrences.

Modeling Bacteria Growth

A number of bacteria doubles every hour and we start with 5 bacteria.

How many bacteria will there be in $n$ hours?

Let $a_n$ be the number of bacteria after $n$ hours

Using techniques from Ch. 2:

$a_0 = 5$,

$a_n = 2 \cdot a_{n-1}$ for $n > 0$

Use an iterative approach to find that $a_n = 5 \cdot 2^n$

Here is a problem we can model with recurrence relations.
Linear Homogeneous Recurrence Relations

A linear homogeneous recurrence relation of degree \( k \) with constant coefficients is a recurrence relation of the form:

\[
a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_k a_{n-k}, \quad (c_1, c_2, \ldots, c_k: \text{real numbers, } c_k \neq 0)
\]

**Linear**: the right-hand side is a sum of previous terms multiplied by some coefficients.

**Homogeneous**: all the terms on the RHS are multiples of previous terms.

**Constant coefficients**: all the coefficients are constants.

**Degree**: number of previous terms in the sequence (\( k \))

Examples of Linear H. Recurrence Relations

Linear homogeneous recurrence relations with constant coefficients:

\[
\begin{align*}
P_n &= (1.11)P_{n-1} \quad ; \text{degree is 1} \\
f_n &= f_{n-1} + f_{n-2} \quad ; \text{degree is 2} \\
a_n &= a_{n-5} \quad ; \text{degree is 5}
\end{align*}
\]

These are not linear homogeneous recurrence relations

\[
\begin{align*}
a_n &= a_{n-1} + (a_{n-2})^2 \quad ; \text{not linear} \\
H_n &= 2H_{n-1} + 1 \quad ; \text{not homogeneous (last term)} \\
B_n &= nB_{n-1} \quad ; \text{non-constant coefficients}
\end{align*}
\]
Characteristic Equation and Roots

Given $a_n = c_1a_{n-1} + ... + c_ka_{n-k}$, look for solutions of the form $a_n = r^n$, where $r$ is a constant.

Plugging into the recurrence, we get: $r^n = c_1r^{n-1} + ... + c_kr^{n-k}$

After dividing both sides by $r^{n-k}$ and rearranging, we get the characteristic equation:

$r^k - c_1r^{k-1} - ... - c_{k-1}r - c_k = 0$

Solutions of this equation are called characteristic roots.

Solving R.R. of Degree Two (Two Roots)

Let $a_n = c_1a_{n-1} + c_2a_{n-2}$ be a recurrence relation, $(c_1, c_2: \text{real numbers})$

Suppose $r^2 - c_1r - c_2 = 0$ (characteristic equation) has two distinct roots $r_1$ and $r_2$ (characteristic roots).

Every sequence of the form:

$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n$ ($n = 0, 1, 2...$); ($\alpha_1$ and $\alpha_2$ constants)

is a solution to the recurrence relation.
Example

What is the solution of the recurrence relation:

\[ a_n = a_{n-1} + 2a_{n-2} \]

with \( a_0 = 2 \) and \( a_1 = 7 \)?

The characteristic equation is \( r^2 - r - 2 = 0 \).
Example

What is the solution of the recurrence relation:

\[ a_n = a_{n-1} + 2 \ a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7 \]?

The characteristic equation is \( r^2 - r - 2 = 0 \).

The characteristic roots are \( r = 2 \) and \( r = -1 \) (quadratic equation).

The theorem says that \( a_n = \alpha_1 2^n + \alpha_2 (-1)^n \) is a solution for some constants \( \alpha_1 \) and \( \alpha_2 \).
Example

Next, use the initial conditions to find $\alpha_1$ and $\alpha_2$:

\[ a_0 = 2 = \alpha_1 + \alpha_2 \]
\[ a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1) \]

Solve these two linear equations in two unknowns and get

$\alpha_1 = 3$ and $\alpha_2 = -1$

The Solution is $a_n = 3 \cdot 2^n - (-1)^n$

Explicit Formula for the Fibonacci Numbers

Find an explicit formula for the Fibonacci numbers:

$f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$
Explicit Formula for the Fibonacci Numbers

Find an explicit formula for the Fibonacci numbers:

\( f_0 = 0, \ f_1 = 1, \) and \( f_n = f_{n-1} + f_{n-2} \)

Characteristic equation is \( r^2 - r - 1 = 0 \)

Characteristic roots are \( r_1 = \frac{1+\sqrt{5}}{2} \) and \( r_2 = \frac{1-\sqrt{5}}{2} \)
Explicit Formula for the Fibonacci Numbers

Find an explicit formula for the Fibonacci numbers:
\[ f_0 = 0, \quad f_1 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \]

Characteristic equation is \[ r^2 - r - 1 = 0 \]

Characteristic roots are \[ r_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2} \]

The theorem says that \[ f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \]
for some constants \( \alpha_1 \) and \( \alpha_2 \)

Use initial conditions to find \( \alpha_1 \) and \( \alpha_2 \)
\[ f_0 = \alpha_1 + \alpha_2 = 0 \]
\[ f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1+\sqrt{5}}{2}\right) = 1 \]

Solve these two equations to get \( \alpha_1 = \frac{1}{\sqrt{5}} \) and \( \alpha_2 = -\frac{1}{\sqrt{5}} \)

The solution is:
\[ f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \]
Solving R.R. of Degree Two (One Root)

Let $a_n = c_1a_{n-1} + c_2a_{n-2}$ be a recurrence relation, ($c_1, c_2 \neq 0$: real numbers)

Suppose $r^2 - c_1r - c_2 = 0$ (characteristic equation) has only one root $r_0$ (characteristic root).

Every sequence of the form:

$a_n = \alpha_1 r_0^n + \alpha_2 nr_0^n$ (n = 0,1,2 ...) ($\alpha_1$ and $\alpha_2$ constants)

is a solution to the recurrence relation.

Example

What is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?
Example

What is the solution of the recurrence relation
\[ a_n = 6a_{n-1} - 9a_{n-2} \]
with initial conditions \( a_0 = 1 \) and \( a_1 = 6 \)?

Characteristic equation is \( r^2 - 6r + 9 = 0 \).

Characteristic root is \( r = 3 \).
Example

What is the solution of the recurrence relation
\( a_n = 6a_{n-1} - 9a_{n-2} \) with initial conditions \( a_0 = 1 \) and \( a_1 = 6 \)?

Characteristic equation is \( r^2 - 6r + 9 = 0 \).

Characteristic root is \( r = 3 \).

Theorem says that \( a_n = \alpha_1 3^n + \alpha_2 n3^n \) for some constants \( \alpha_1 \) and \( \alpha_2 \).

Use initial conditions to find \( \alpha_1 \) and \( \alpha_2 \):

\( a_0 = 1 = \alpha_1 \)

\( a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3 \)

Solve these two equations to get \( \alpha_1 = 1 \) and \( \alpha_2 = 1 \)

The solution is \( a_n = 3^n + n3^n \).
Solving R.R. of Degree k, Distinct Roots

Let \( a_n = c_1a_{n-1} + c_2a_{n-2} \) be a recurrence relation \((c_1, \ldots, c_k):\) real numbers.

Suppose \( r^k - c_1r^{k-1} - \ldots - c_k = 0 \) (characteristic equation) has \( k \) roots \( r_1, \ldots, r_k \) (characteristic roots).

Every sequence of the form:
\[
a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n
\]
\((n = 0,1,2 \ldots)\) with \( \alpha_1, \ldots, \alpha_k \) constants
is a solution to the recurrence relation.

Example

What is the solution to the recurrence relation
\[
a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}
\]
with initial conditions \( a_0 = 2, a_1 = 5, \) and \( a_2 = 15 \)
Example

What is the solution to the recurrence relation
\[ a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \]
with initial conditions \( a_0 = 2 \), \( a_1 = 5 \), and \( a_2 = 15 \)

Characteristic equation is \( r^3 - 6r^2 + 11r - 6 = 0 \)

Characteristic roots are \( r = 1 \), \( r = 2 \), and \( r = 3 \)
Example

What is the solution to the recurrence relation

\[ a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \]

with initial conditions \( a_0 = 2, \ a_1 = 5, \) and \( a_2 = 15 \)

Characteristic equation is \( r^3 - 6r^2 + 11r - 6 = 0 \)

Characteristic roots are \( r = 1, \ r = 2, \) and \( r = 3 \)

Theorem says that \( a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n \)

Example

Use the initial conditions to find \( \alpha_1, \alpha_2 \) and \( \alpha_3 \):

\[ a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3 \]
\[ a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 \]
\[ a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9 \]

Solve these simultaneous equations to get \( \alpha_1 = 1 \) and \( \alpha_2 = -1 \) and \( \alpha_3 = 2 \)

The solution is \( a_n = 1 - 2^n + 2 \cdot 3^n \)
End