Sequences and Sums

Chapter 2
Sequences

A person has two parents, four grandparents, eight great-grandparents, and so forth ...

\[ 2, 4, 8, 16, 32, 64, 128, \ldots \]

To express a pattern, suppose that each generation is labeled by an integer giving its position in the row.

<table>
<thead>
<tr>
<th>Position in the row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of ancestors</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128…</td>
</tr>
</tbody>
</table>

Each successive number doubles in size, or is a power of 2

For a general value of \( k \), let \( a_k \) be the number of ancestors in the \( k \)th generation back.

\[ a_1 = 2^1, \quad a_2 = 2^2, \quad a_3 = 2^3, \quad a_4 = 2^4 \ldots \]

In general, we note that the pattern is \( a_k = 2^k \)

This pattern is called a sequence.
Sequence - Definition

A sequence is a function whose domain is either:

- All the integers between two given integers, or
- All the integers greater than or equal to a given integer

Sequence Notation

We typically represent a sequence as a set of elements written in a row. In the sequence denoted

\[ a_m, a_{m+1}, a_{m+2}, \ldots, a_n, \]

where each individual element \( a_k \) (read “a sub k”) is called a term.

1. \( k \) in \( a_k \) is a subscript or index
2. \( m \) is the subscript of the initial term
3. \( n \) is the subscript of the final term.
Activity 1.0

1. Assume we have the following sequence of numbers:
   \[3, 5, 7, 9, 11, 13, 15, \ldots\]
   Can we determine a general term of the sequence \(a_k\)?

2. Write the first four terms of the following sequence:
   \[a_k = \frac{k}{10 + k}, \forall \text{ integers } k \geq 1\]

Geometric progression

A geometric progression is a sequence of the form:
\[a, ar, ar^2, \ldots, ar^n \ldots\]
where the initial term \(a\), and the common ratio \(r\) are real numbers.

e.g.
\[c = 2 \cdot 5^n\]
\[2, 10, 50, 250, 1250 \ldots\]
Arithmetic progression

An arithmetic progression is a sequence of the form:
a, a + d, a + 2d, ..., a + nd, ...
where the initial term $a$, and the common difference $d$
are real numbers.

e.g.
t = 7 – 3n
7, 4, 1, -2 ...

Recurrence relation
Fibonacci numbers

The Fibonacci numbers are the numbers in the following integer sequence.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Recurrence relation

A recurrence relation for the sequence \{a_n\} is an equation that expresses \(a_n\) in terms of one or more of the previous terms of the sequence.
Fibonacci numbers

Pseudocode

1. Set the first number to 0
2. Set the second number to 1
3. Add the previous two numbers to get the current number
4. Repeat step 3 until done

Initial condition: \( f_0 = 0 \) and \( f_1 = 1 \)

Recurrence relation: \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \)

Example

We have the following recurrence relation:

\[ a_k = 2a_{k-1} + k, \text{ for all integers } k \geq 2 \]

Given the initial condition \( a_1 = 1 \)

What are the first four terms in this sequence?

\( k = 2 \): \( a_2 = 2a_1 + 2 = 2(1) + 2 = 4 \)
\( k = 3 \): \( a_3 = 2a_2 + 3 = 2(4) + 3 = 8 + 3 = 11 \)
\( k = 4 \): \( a_4 = 2a_3 + 4 = 2(11) + 4 = 22 + 4 = 26 \)
Let $a_0, a_1, a_2 \ldots$ be defined by the formula $a_n = 3n + 1$ for all integers $n \geq 0$.

Identify the first 5 terms:

- $(n = 0), a_0 = 3(0) + 1 = 1$
- $(n = 1), a_1 = 3(1) + 1 = 4$
- $(n = 2), a_2 = 3(2) + 1 = 7$
- $(n = 3), a_3 = 3(3) + 1 = 10$
- $(n = 4), a_4 = 3(4) + 1 = 13$

Write the sequence $a_n = 3n + 1$ as a recurrence relation.
We are given a sequence $t_0, t_1, t_2, \ldots$ that is defined by the formula $t_n = 2 + n$ for all integers $n \geq 0$.

Write a recurrence relation for this sequence using $t_{k-1}$ and $t_{k-2}$.

Let’s switch order!

A solution of a recurrence relation is a sequence $\{a_n\}$ whose terms satisfies that recurrence relation.
Solution of a recurrence relation

Prove that the sequence of factorials $a_n = n!$ is a solution to the recurrence relation $a_n = n \cdot a_{n-1}$

Solution of a recurrence relation

The sequence of factorials satisfies the recurrence relation $a_n = n \cdot a_{n-1}$; therefore, $a_n = n!$ is a solution to the recurrence relation $a_n = n \cdot a_{n-1}$
Activity 2.0

Determine whether the sequence \( a_n = 3n \) is a solution of the recurrence relation \( a_n = 2a_{n-1} - a_{n-2} \) for \( n \geq 2 \)

Activity 3.0

Determine whether the sequence \( a_n = 2^n \) is a solution to the recurrence relation \( a_n = 2a_{n-1} - a_{n-2} \) for \( n \geq 0 \)
Summation Notation

For a sequence $a_k = 2k+1$

What is the sum of all $a_k$ for $k = 1$ to $3$?

The first three terms are: 3, 5, 7
The sum of these terms is $3 + 5 + 7 = 15$

What is the sum of all $a_k$ for $k=1$ to $n$?

$\text{sum} = 3 + 5 + 7 + \ldots + a_n$

A shorthand notation is to use the summation notation

Given integers $m$ and $n$ where $m \leq n$, the symbol:

$$\sum_{k=m}^{n} a_k$$

is the sum of all the terms $a_m, a_{m+1}, \ldots, a_n$
Summation Notation

We can also express the summation as:

\[ \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n \]

where the right-hand side (RHS) of the equation is called the expanded form of the sum.

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Revisiting our previous example, \( a_k = 2k+1 \), what is the sum of all \( a_k \) for \( k=1 \) to \( n \)?

For our example, the summation notation is:

\[ \sum_{k=m}^{n} a_k = \sum_{k=1}^{n} (2k + 1) \]
Summation Notation Expanded Form

The expanded form of the summation notation for our example is the following:

\[ \sum_{k=1}^{n} (2k + 1) = (2(1) + 1) + (2(2) + 1) + (2(3) + 1) + \cdots + (2n + 1) \]

\[ = 3 + 5 + 7 + \cdots + (2n + 1) \]

Separating Off the Final Term

We can also write our sum as the following:

\[ \sum_{k=1}^{n} (2k + 1) = \sum_{k=1}^{n-1} (2k + 1) + (2n + 1) \]

This is known as separating off the final term.
Product Notation

Given integers\( m \) and\( n \) where\( m \leq n \), the symbol:

\[
\prod_{k=m}^{n} a_k
\]

is the product of all the terms\( a_m, a_{m+1}, \ldots, a_n \)

Properties of Summations and Products

\[
\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)
\]

\[
c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k
\]

\[
\left( \prod_{k=m}^{n} a_k \right) \cdot \left( \prod_{k=m}^{n} b_k \right) = \prod_{k=m}^{n} (a_k \cdot b_k)
\]
Change of Variable

\[ \sum_{k=1}^{3} (k - 1) \]

Can we simplify with change of variable?

What if we let \( j = k - 1 \)?

What are new upper and lower limits of summation?

(k = 1): \( j = k - 1 = 1 - 1 = 0 \)

(k = 3): \( j = k - 1 = 3 - 1 = 2 \)

This becomes:

\[ \sum_{k=1}^{3} (k - 1) = \sum_{j=0}^{2} j \]
End