Instructions:

1. You are allowed to use one sheet of notes (8.5 x 11 - one side)

2. This is a closed book exam. Do not confer with any other person. Do not use any computer equipment.

3. Show your work. Partial credit will be given. Grading will be based on correctness, clarity and neatness.

4. I suggest that you read the whole exam before beginning to work any problem. Budget your time wisely.

**DO NOT BEGIN THE EXAM UNTIL INSTRUCTED TO DO SO. GOOD LUCK!**

Please sign the academic integrity statement:

“On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work. In particular, I certify that I have not received or given any assistance that is contrary to the letter or the spirit of the guidelines for this exam.”

Signature: _______________________________
Mathematical Induction Template:
Let \( P(n) := \text{PREDICATE}, \) when \( n \geq BASE\ CASE \)

\textbf{Basis Step:} \( P(BASE\ CASE) \)

WORK TO SHOW THE BASE CASE

\( \therefore P(BASE\ CASE) \) holds.

\textbf{Inductive Step:} \( P(k) \implies P(k + 1) \)
Assume \( P(k) \) for some \( k \geq BASE\ CASE : \text{STATE} P(k) \)
Show \( P(k + 1) : \text{STATE} P(k + 1) \)

USE THE INDUCTIVE HYPOTHESIS \hspace{1cm} \text{(by inductive hypothesis)}
WORK TO SHOW \( P(k + 1) \)

\( \therefore P(k) \implies P(k + 1) \) holds for \( k \geq BASE\ CASE \).

\( \therefore P(n) \) holds for all \( n \geq BASE\ CASE \) by mathematical induction.
1. (10 points) **Sequences and recurrence relations**
   A client deposits $100 in a bank account. Every year the bank account yields a compound interest of 20%.

   **(a) (6 points)** Set up a recurrence relation for the amount in the account after \( n \) years.
   \[
   a_0 = 100$
   \]
   \[
   a_n = a_{n-1} + 20\% a_{n-1} = a_{n-1} + 0.2a_{n-1} = a_{n-1}(1.2)
   \]

   **(b) (4 points)** How much money will the account contain after 3 years?
   \[
   a_0 = 100$
   \]
   \[
   a_1 = 100 \times 1.2 = 120$
   \]
   \[
   a_2 = 120 \times 1.2 = 144$
   \]
   \[
   a_3 = 144 \times 1.2 = 172.8$
   \]
2. (20 points) Sequences and recurrence relations

(a) (6 points) Show that the sequence $a_n = 2(-4)^n + 3$ is a solution to the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$

Substitute $a_{n-1}$ and $a_{n-2}$ in the sequence $a_n$

(b) (7 points) Find a recurrence relation satisfied by the sequence: $a_n = n^2 + n$

$a_n = a_{n-1} + 2n, a_0 = 0$

(c) (7 points) Find a recurrence relation satisfied by the sequence: $a_n = n + (-1)^n$

$a_n = a_{n-1} + 1 + 2(-1)^n, a_0 = 1$
3. (20 points) Algorithms

(a) (6 points) Use bubble sort to sort 8, 3, 7, 5 in increasing order. Show the list obtained at each step.

1. \[\begin{array}{cccc}
8 & 3 & 7 & 5 \\
3 & 8 & 7 & 5 \\
3 & 7 & 8 & 5 \\
3 & 7 & 5 & 8
\end{array}\]

2. \[\begin{array}{cccc}
3 & 7 & 5 & 8 \\
3 & 7 & 5 & 8 \\
3 & 5 & 7 & 8
\end{array}\]

3. \[\begin{array}{cccc}
3 & 5 & 7 & 8 \\
3 & 5 & 7 & 8
\end{array}\]

(b) (6 points) List all the steps used to search for 9 in the sequence: 1, 3, 4, 5, 6, 8, 9, 11 using a binary search.

Search from index: 1 to index: 8
midpoint = \(\left\lfloor \frac{1+8}{2} \right\rfloor = \left\lfloor \frac{9}{2} \right\rfloor = 4\)
5 < 9

Search from index: 5 to index: 8
midpoint = \(\left\lfloor \frac{5+8}{2} \right\rfloor = \left\lfloor \frac{13}{2} \right\rfloor = 6\)
8 < 9

Search from index: 7 to index: 8
midpoint = \(\left\lfloor \frac{7+8}{2} \right\rfloor = 7\)
9 = 9, found!
(c) (8 points) Write in English an algorithm that takes as input a list of \( n \) integers and produces as output the largest sum obtained by adding an integer in the list to the one following it.

DO NOT write code or pseudocode! describe your algorithm in plain English.

(a) Add the first number to the second number and set the sum to the temporary maximum
(b) Add the second number to the number following it. If the new sum is greater than the temporary maximum, replace the temporary maximum with the new sum.
(c) Repeat step 2 until the number before the last number
(d) The largest sum is the temporary maximum obtained at the end of step 3.

4. (14 points) Growth of functions

(a) (7 points) Give a big-O estimate for the function \( f(x) = (x! + 2^x)(x^3 + \log(x^2 + 1)) \)

\( f(x) \) is \( O(x!x^3) \)

(b) (7 points) Give a big-O estimate for the function \( f(x) = (5x \log x + x^2)(x^3 + 6) \)

\( f(x) \) is \( O(x^5) \)
5. (10 points) Analysis of algorithm efficiency

For the code segment below, assume that \( n \) is a positive integer.

Compute the number of additions, multiplications, and comparisons that must be performed when the code segment is executed.

```plaintext
for (i=1; i <= n; i++) {
    for (j=1; j <= 2*n; j++) {
        a = (2*n) + (i*j);
    }
}
```

Line 1: (1 add, 1 comp), \( n \) times
Line 2: (1 add, 1 mul, 1 comp), \( n \times 2 \times n = 2n^2 \) times
Line 3: (2 mul, 1 add), \( n \times 2 \times n = 2n^2 \) times

additions: \( n + 2n^2 + 2n^2 = n + 4n^2 \)
multiplications: \( 2n^2 + 2n^2 + 2n^2 \)
comparisons: \( n + 2n^2 \)
6. (8 points) Proof by mathematical induction

Prove by mathematical induction that:

\[ 1^3 + 2^3 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \]
for all integers \( n \geq 1 \)

**Basis step:** show \( P(1) \)

\[ 1^3 = 1 \]
\[ \left( \frac{1(1+1)}{2} \right)^2 = \left( \frac{2}{2} \right)^2 = 1 \]

\[ \therefore P(1) \text{ holds.} \]

**Inductive step:** show \( P(k) \to P(k+1) \)

Inductive hypothesis: \( P(k) = 1^3 + 2^3 + \ldots + k^3 = \left( \frac{k(k+1)}{2} \right)^2 \)

\[ P(k+1) = 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2 \]

According to the inductive hypothesis:

\[ 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3 \]
\[ = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \]
\[ = \frac{(k+1)^2(k^2 + 4(k+1))}{4} \]
\[ = \frac{(k+1)^2(k+2)^2}{4} \]
\[ = \left( \frac{(k+1)(k+2)}{2} \right)^2 \]

\[ \therefore P(k) \to P(k+1) \]

\[ \therefore P(n) \text{ is true.} \]
7. (8 points) Proof by mathematical induction

Prove by mathematical induction that:

\( n^3 + 2n \) is a multiple of 3 whenever \( n \) is a positive integer.

**Proof:** Let \( P(n) := 3|(n^3 + 2n) \), when \( n \geq 1 \)

**Basis Step:** \( P(1) \)

\[ 1^3 + (2 \cdot 1) = 3 \]
\[ 3|3 \]
\[ \therefore P(1) \text{ holds.} \]

**Inductive Step:** \( P(k) \implies P(k + 1) \)

Assume \( P(k) \) for some \( k \geq 1 : 3|(k^3 + 2k) \)

Show \( P(k + 1) : 3|((k + 1)^3 + 2(k + 1)) \)

\[ (k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1) \]
\[ 3|(k^3 + 2k) + 3(k^2 + k + 1) \]
\[ \therefore P(k) \implies P(k + 1) \text{ holds for } k \geq 1. \]

\[ \therefore P(n) \text{ holds for all } n \geq 1 \text{ by mathematical induction.} \]
8. (10 points) Proof by mathematical induction

(a) (4 points) Conjecture a formula for this sum by examining the values of this expression for small integers n:

\[ \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)} \]

(b) (6 points) Prove the formula that you conjectured in part (a) by mathematical induction.
Solution Let \( S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} \). Then

\[
S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \quad S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}, \ldots
\]

and we conjecture that \( S_n = \frac{n}{n+1} \). We prove this via induction. Note that we have already shown this to be true for \( n = 1 \). Suppose \( S_m = \frac{m}{m+1} \) for \( m \in \mathbb{N} \). Then

\[
S_{m+1} = \sum_{k=1}^{m+1} \frac{1}{k(k+1)} = \sum_{k=1}^{m} \frac{1}{k(k+1)} + \frac{1}{(m+1)(m+2)}
\]

and so

\[
S_{m+1} = S_m + \frac{1}{(m+1)(m+2)}.
\]

It follows from our induction hypothesis that

\[
S_{m+1} = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} = \frac{m(m+2) + 1}{(m+1)(m+2)} = \frac{m+1}{m+2}.
\]

This completes the proof.
9. (2 points) Extra credit

Prove by Strong Induction that if $n$ is an integer greater than 1, then $n$ is divisible by a prime number.

Refer to lecture slides examples on Strong Induction.