The number system

Number system and Binary numbers
What is the number system?

The number system is a writing system for expressing numbers.

It is composed of a collection of symbols used to represent small numbers, together with a system of rules for representing larger numbers.

<table>
<thead>
<tr>
<th>Number Systems</th>
<th>Base</th>
<th>Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary</td>
<td>2</td>
<td>0 1</td>
</tr>
<tr>
<td>Octal</td>
<td>8</td>
<td>0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>Decimal</td>
<td>10</td>
<td>0 1 2 3 4 5 6 7 8 9</td>
</tr>
<tr>
<td>Hexadecimal</td>
<td>16</td>
<td>0 1 2 3 4 5 6 7 8 9 A B C D E F</td>
</tr>
</tbody>
</table>

Example of a 32-bit instruction

LDR <Rd>, [<Rn>, #<immed_5> * 4]

cond  op  reserved
code  code

32 bit ARM LDR
Example of a 32-bit instruction

**Instruction format (machine language)**

- **Machine Language**
  - Computers do not understand “add R8, R17, R18”
  - Instructions are translated to machine language (1s and 0s)

- **Example:**
  
  ```
  add R8, R17, R18 →
  00000010 00110010 01000000 00100000
  ```

---

**Number system: unary**

What is the first numeral system we learn?
What is the most natural numeral system?

The unary uses one symbol.
This is the simplest way of counting
Number system: Roman

Numbers in the Roman system are represented by combinations of letters from the Latin alphabet.

Roman numerals used today, have seven symbols, each with a fixed integer value:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>I</th>
<th>V</th>
<th>X</th>
<th>L</th>
<th>C</th>
<th>D</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>50</td>
<td>100</td>
<td>500</td>
<td>1,000</td>
</tr>
</tbody>
</table>

Number system: base

A radix, or base, is the number of unique digits, including zero, used to represent numbers in a positional numeral system.

![image credit: http://thepictutor.com/decimal.html](http://thepictutor.com/decimal.html)
Number system: base

Generalized form of positional systems in Base B:
Commonly used numeral systems include:

- **Binary** (Base 2): 0, 1
- **Octal** (Base 8): 0, 1, 2, 3, 4, 5, 6, 7
- **Decimal** (Base 10): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- **Hexadecimal** (Base 16): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Consider a number N with n+1 digits:

\[ N = a_n \times a_{n-1} \times a_{n-2} \times ... \times a_0 \]

most significant digit

least significant digit

\[ N_B = a_n \times B^n + a_{n-1} \times B^{n-1} + a_{n-2} \times B^{n-2} + ... + a_1 \times B^1 + a_0 \times B^0 \]
Number system: base

The Egyptians had a base-10 system of hieroglyphs for numerals. They separate symbols for one unit, one ten, one hundred, one thousand, one ten thousand, one hundred thousand, and one million.

<table>
<thead>
<tr>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
<th>100000</th>
<th>1000000</th>
</tr>
</thead>
</table>

Egyptian numeral hieroglyphs

e.g. The number 276 in hieroglyphs, requires fifteen symbols:
Number system: base

<table>
<thead>
<tr>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
<th>100000</th>
<th>10^5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

Egyptian numeral hieroglyphs

e.g. The number 4622 in hieroglyphs, requires these symbols:

Number system: Binary numbers

Conversion of a number from binary to decimal:

\[ N_2 = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{array} \]

The value of \( N_2 \) in the Decimal Base \( N_{10} \) is:

\[ N_{10} = 1 \times 2^9 + 0 \times 2^8 + 0 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \]

\[ = 537 \]
Number system: Binary numbers

Conversion of a number from decimal to binary:

To convert a decimal number to binary:

1. First, subtract the largest possible power of two
2. Keep subtracting the next largest possible power of 2 from the remainder, marking 1s in each column where this is possible and 0s where it is not.

Converting the decimal (base 10) number 86 to binary (base 2)

64 is the largest power of 2 that goes into 86. The result is:

<table>
<thead>
<tr>
<th>64</th>
<th>32</th>
<th>16</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
</table>
86 - 64 is 22. 32 is larger than 22, so a 0 is placed in the bit for the value 32. 16 is less than 22, so a 1 is placed in the bit for the value 16.

<table>
<thead>
<tr>
<th>64</th>
<th>32</th>
<th>16</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
22 - 16 is 6. 8 is larger than 6. The bits 4+2 equal 6 so each of those bits become 1

<table>
<thead>
<tr>
<th>64</th>
<th>32</th>
<th>16</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
This 86 in decimal is 1010110 in binary. \(86_{10} = 1010110_2\)
Number system: conversion int()

Conversion using Python functions.

We can use the int() built-in function to convert any base k number to decimal.

The syntax of int() method is: int(x, base)

```
>>> # binary conversion to integer
>>> print(f'1110 is integer: {int('1110', 2)}')
1110 is integer: 14

>>> # hexadecimal conversion to integer
>>> print(f'AB is integer: {int('ABC', 16)}')
AB is integer: 2748
```

The syntax of int() method is: int(x, base)
x: number or string to convert

base: base of the number in x

```
>>> int('241103')
241103
>>> int('241103', 10)
241103
>>> int('100101', 2)
37
>>> int('9401BA', 16)
9699770
```
Number system: Binary numbers \texttt{bin()}

The \texttt{bin()} method takes one parameter:

\textbf{input}: the integer number to convert to binary

\textbf{output}: a binary string

If the input is not an integer, \texttt{bin()} raises a \texttt{TypeError} exception.

```python
> numbers = [25, 16, '128']
> for n in numbers:
>     try:
>         print(bin(n))
>     except TypeError:
>         print("Incorrect type for binary conversion")

0b11001
0b10000
Incorrect type for binary conversion
```

Number system: Binary numbers

Conversion using Python functions.

You can use the \texttt{bin()} built-in function to convert a decimal number to binary. \texttt{bin()} outputs the binary number as a string.

```python
>>> # The decimal number 50 in binary
>>> # '0b' is the prefix for a binary number
>>> bin(50)
'0b110010'
>>> bin(357)
'0b101100101'
>>> # ignoring the prefix '0b' in the output of \texttt{bin()}
>>> bin(357)[2:]
'101100101'
```
Number system: hexadecimal numbers

Conversion using Python functions.

The `hex()` function converts an integer to the corresponding hexadecimal number and returns it.

```python
numbers = [25, 16, 128]
for n in numbers:
    try:
        print(hex(n))
    except TypeError:
        print("Incorrect type for binary conversion")
```

Number system: Binary addition

Rules for binary addition

0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 10

for 1 + 1, we write down a zero in the right-most column and carry over a one to the next column.
Number system: Binary addition

```python
# Convert decimal 19 to binary number '0b10011'
print(bin(19))

# 0b on the previous line means the result is binary
print(int("10011", 2))

# the 4 binary addition rules
rule1 = bin(int("0", 2) + int("0", 2)) # 0 + 0 = 0
rule2 = bin(int("0", 2) + int("1", 2)) # 0 + 1 = 1
rule3 = bin(int("1", 2) + int("0", 2)) # 1 + 0 = 1
rule4 = bin(int("1", 2) + int("1", 2)) # 1 + 1 = 10

# Adding binary numbers 10101 and 110 results in 11011
number = bin(int("10101", 2) + int("110", 2))
print(number)
```

Number system: bitwise operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>~</td>
<td>Binary NOT</td>
</tr>
<tr>
<td></td>
<td>Each bit of the output is 1 if the bit of x is 0, otherwise it is 0.</td>
</tr>
<tr>
<td>&amp;</td>
<td>Binary AND</td>
</tr>
<tr>
<td></td>
<td>Each bit of the output is 1 if the bit of x AND the bit of y is 1, otherwise it's 0.</td>
</tr>
<tr>
<td></td>
<td>Binary OR</td>
</tr>
<tr>
<td></td>
<td>Each bit of the output is 0 if the bit of x AND the bit of y is 0, otherwise it's 1.</td>
</tr>
<tr>
<td>^</td>
<td>Binary XOR</td>
</tr>
<tr>
<td></td>
<td>Each bit of the output is 1 if the bit of x is the one complement of the bit of y, otherwise it's 0.</td>
</tr>
<tr>
<td>&lt;&lt;</td>
<td>Binary Left Shift</td>
</tr>
<tr>
<td>x &lt;&lt; y returns x with the bits shifted to the left by y places. Performs a multiplication of x by $2^y$.</td>
<td></td>
</tr>
<tr>
<td>&gt;&gt;</td>
<td>Binary Right Shift</td>
</tr>
<tr>
<td>x &gt;&gt; y returns x with the bits shifted to the right by y places. Performs a floor division of x by $2^y$.</td>
<td></td>
</tr>
</tbody>
</table>
Bitwise operator: NOT (~)

The NOT of a binary number is the value obtained by inverting all the bits in the binary number.

- We flip 0 to 1
- We flip 1 to 0

Truth table:

---

Bitwise operator: NOT (~)

How to read the NOT of a binary number?

In programming we read the NOT of a binary number in 2’s complement format.

2's complement is a method of representing signed integers on computers.

To get the 2's complement of a binary number:

1. Invert all the bits.
2. Add 1 to the inverted bits.
Bitwise operator: NOT (~)

e.g.: What is \(~12\) in binary and in decimal?

Convert 12 to binary: \(12 = 1100\)

Flip to bits: \(~12 = 0011\)

To read 0011 in decimal, perform the 2’s complement of 0011.

<table>
<thead>
<tr>
<th>2’s complement of 0011</th>
</tr>
</thead>
<tbody>
<tr>
<td>(~12:)</td>
</tr>
<tr>
<td>0011</td>
</tr>
<tr>
<td>flip the bits:</td>
</tr>
<tr>
<td>1100</td>
</tr>
<tr>
<td>add 1:</td>
</tr>
<tr>
<td>1101</td>
</tr>
</tbody>
</table>

\(1101 = 13\)

Answer: \(~12 = 0011 = -13\)
Bitwise operator: AND (&)

The AND of a set of operands is 1 if and only if all of the operands are 1’s.

Truth table:
Bitwise operator: OR (|)

The OR of a set of operands is \(1\) if at least one of the operands is a \(1\).

Truth table:
Bitwise operator: XOR (^)

The XOR of a set of operands is 1 if both the operands differ.

Truth table:
Left shift: ($\ll$)

A logical shift is a bitwise operation that shifts all the bits of its operand.

$n \ll x$ is $n$ with the bits shifted to the left by $x$ places.

Note: the left shift performs a multiplication of $n$ by $2^x$. 
Right shift: (>>)

A logical shift is a bitwise operation that shifts all the bits of its operand.

\[ n >> x \] is \( n \) with the bits shifted to the right by \( x \) places.

Note: the right shift performs a \textit{floor division} of \( n \) by \( 2^x \).
Number system: bitwise operators